

On the strong connectivity problem

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We show that the strong connectivity problem is solvable in polynomial time in case each value α in the distance matrix with $0 < \alpha < \infty$ is contained in a submatrix of form $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ (up to permuting rows or columns), thus extending a result of Lucchesi.

The *strong connectivity augmentation problem* is:

(1) *given*: a directed graph $G = (V, A)$, a length function $l: V \times V \rightarrow \mathbf{Z}_+$ and an integer B ,

find: a set $A' \subseteq V \times V$ so that the graph $(V, A \cup A')$ is strongly connected and so that $\sum_{a \in A'} l(a) < B$.

(cf. Garey and Johnson [3]). This problem is easily seen to be *NP*-complete, since the problem of finding a Hamiltonian cycle in a directed graph (V, A'') is reducible to (1): just take $A := \emptyset$, $l: V \times V \rightarrow \mathbf{Z}_+$ defined by:

$$(2) \quad \begin{aligned} l(u, v) &:= 0 \text{ if } (u, v) \in A'', \\ &:= 1 \text{ if } (u, v) \notin A'', \end{aligned}$$

and $B := 1$ (cf. Eswaran and Tarjan [1]).

In fact the *traveling salesman problem*:

(3) *given*: a length function $l': V \times V \rightarrow \mathbf{Z}_+$ and an integer B' ,

find: a Hamiltonian cycle of length less than B'

is a direct special case of (1) (take $A := \emptyset$, $l(u, v) := l'(u, v) + B'$ and $B := B' |V| + B'$).

Another application of the strong connectivity augmentation problem is the planar feedback arc set problem (see below).

The strong connectivity augmentation problem is trivially equivalent to the *strong connectivity problem*:

(4) *given*: a length function $l: V \times V \rightarrow \mathbf{Z}_+ \cup \{\infty\}$ and an integer B ,

find: a subset $A' \subseteq V \times V$ so that (V, A') is strongly connected and so that $\sum_{a \in A'} l(a) < B$.

Indeed, (4) is just the case $A = \emptyset$ in (1). Conversely, (1) can be reduced to (4) by resetting $l(a) := 0$ whenever $a \in A$. Allowing $l(a) = \infty$ in (4) is

irrelevant: we could replace any ∞ by the value B .

We may assume in (4) without loss of generality that for all $i, j, k \in V$:

- (5) (i) $l(i, i) = 0$
- (ii) if $l(i, j) = 0$ and $l(j, k) = 0$ then $l(i, k) = 0$.

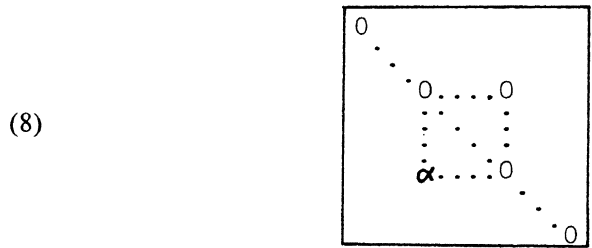
It was shown by Lucchesi [5] (cf. Frank [2] and Lucchesi and Younger [6]) that the strong connectivity problem (4) is solvable in polynomial time if the following condition on the length function holds:

- (6) for each $i, j \in V$: if $0 < l(i, j) < \infty$ then $l(j, i) = 0$.

Equivalently, the strong connectivity augmentation problem is solvable in polynomial time if:

- (7) for each $i, j \in V$: if $0 < l(i, j) < B$ then $(j, i) \in A$.

So problem (1) is solvable in polynomial time if in the distance table we have that for each value α with $0 < \alpha < \infty$, the symmetric value is equal to 0:



Lucchesi showed that this implies a polynomial-time algorithm for the following *feedback arc set problem*, in case G is planar:

- (9) *given*: a directed graph $G = (V, A)$, a length function $l: A \rightarrow \mathbb{Z}_+$ and an integer B ,
- find*: a subset $A' \subseteq A$ so that (V, A') is acyclic and so that $\sum_{a \in A'} l(a) > B$.

In general, this problem is *NP*-complete (Karp [4]).

To see that (9) is solvable in polynomial time if G is planar, we consider the planar dual graph $G^* = (F, A^*)$ of G , directed in such a way that each arc of G crosses its dual arc in G^* 'from left to right':



(where the uninterrupted arrow is an arc of G , and the interrupted arc is the dual arc in G^*). Define for each pair $(f, g) \in F \times F$:

$$(11) \quad \left. \begin{array}{l} l^*(f,g) := 0 \\ l^*(g,f) := l(a) \\ l^*(f,g) := \infty \end{array} \right\} \quad \begin{array}{l} \text{if } (f,g) = a^* \in A^*, \text{ where} \\ \{a^* \text{ is the dual arc of } a \in A, \\ \text{for all other pairs } (f,g). \end{array}$$

Let $B^* := \sum_{a \in A} l(a) - B$. Then for each subset A' of A one has:

$$(12) \quad (V, A') \text{ is acyclic} \Leftrightarrow (F, A^* \cup [(A \setminus A')^*]^{-1}) \text{ is strongly connected}$$

(here C^{-1} denotes the set of inverse arcs of C). Moreover,

$$(13) \quad \sum_{a \in A'} l(a) > B \Leftrightarrow \sum_{(g,f) \in [(A \setminus A')^*]^{-1}} l^*(g,f) < B^*.$$

This reduces the planar feedback arc set problem to the strong connectivity problem satisfying (6). Hence it is solvable in polynomial time.

Lucchesi's algorithm can also be used in a branch and bound method to solve the general strong connectivity problem. Typically, during the branching process, a node of the tree is labeled by a set R of 'required' arcs and a set F of 'forbidden' arcs. That is, the node only considers those subsets A' of $V \times V$ for which $R \subseteq A' \subseteq (V \times V) \setminus F$ and for which (V, A') is strongly connected. So the bound corresponding to the node should be a lower bound on the minimum length of these subsets A' .

In order to find such a bound, we can assume that R is reflexive (i.e., $(i,i) \in R$ for all i) and transitive (i.e., if (i,j) and (j,k) belong to R , then $(i,k) \in R$). Moreover, we can reset

$$(14) \quad \begin{array}{ll} l(a) := 0 & \text{if } a \in R, \\ l(a) := \infty & \text{if } a \in F. \end{array}$$

If after this resetting, Lucchesi's condition:

$$(15) \quad \text{for all } i, j \in V: \text{ if } 0 < l(i,j) < \infty \text{ then } l(j,i) = 0$$

is satisfied, Lucchesi's algorithm gives us the exact minimum value (instead of a lower bound) in polynomial time. This suggests that in our branching strategy, we should strive for a situation where (15) holds. That is, for choices of R and F satisfying:

$$(16) \quad \text{for all } i, j \in V: (i,j) \in R, \text{ or } (j,i) \in R, \text{ or both } (i,j) \in F \text{ and } (j,i) \in F.$$

We show that the strong connectivity problem can also be solved in polynomial time if we weaken Lucchesi's condition (15) to:

$$(17) \quad \text{for all } i, j \in V: \text{ if } 0 < l(i,j) < \infty \text{ then } \exists i', j' \in V \text{ with} \\ l(i,i') = l(j',i') = l(j',j) = 0.$$

This is indeed weaker than Lucchesi's condition, since if (15) holds we can take $i' = i$ and $j' = j$ in (17).

Condition (17) means that in the distance table we have that any value with $0 < \alpha < \infty$ is part of a 2×2 -matrix $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ as in:

$$(18) \quad \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \alpha & 0 \\ \hline \end{array}$$

So the difference with Lucchesi's condition is that the diagonal elements of $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$ need not be diagonal elements of the distance matrix.

THEOREM. *The strong connectivity problem is solvable in polynomial time if (17) is satisfied.*

PROOF. Let l satisfy (18). We may assume furthermore that $l(i,i) = 0$ for all $i \in V$, and if $l(i,j) = l(j,k) = 0$ then $l(i,k) = 0$ for all $i, j, k \in V$.

Suppose now that $0 < l(i,j) < \infty$ for some $i, j \in V$ while $l(j,i) \neq 0$. By (17) there exist $i', j' \in V$ so that $l(j', i) = l(j', i') = l(j, i') = 0$. We introduce two new points, i'' and j'' say. Let $\bar{V} := V \cup \{i'', j''\}$, and

$$(19) \quad \begin{aligned} \bar{l}(a,b) &:= l(a,b) \quad \text{if } a, b \in V, (a,b) \neq (i,j), \\ \bar{l}(i,j) &:= \infty \\ \bar{l}(i,i'') &:= \bar{l}(i'',i') := \bar{l}(j',j'') := \bar{l}(j',j'') := \bar{l}(j'',j) := 0, \\ \bar{l}(a,b) &:= \infty \quad \text{for all other } a, b \in \bar{V}. \end{aligned}$$

We show that the strong connectivity problem for \bar{V}, \bar{l} is equivalent to that for V, l . First, let A be a minimum length subset of $V \times V$ with (V, A) strongly connected. Let:

$$(20) \quad \begin{aligned} \bar{A} &:= A \cup \{(i,i''), (i'',i'), (j',i''), (j',j''), (j'',j)\} && \text{if } (i,j) \notin A, \\ \bar{A} &:= (A \setminus \{(i,j)\}) \cup \{(i,i''), (i'',i'), (j',i''), (j',j''), (j'',j), (i'',j'')\} && \text{if } (i,j) \in A. \end{aligned}$$

Clearly,

$$(21) \quad \sum_{a \in A} l(a) = \sum_{a \in \bar{A}} \bar{l}(a).$$

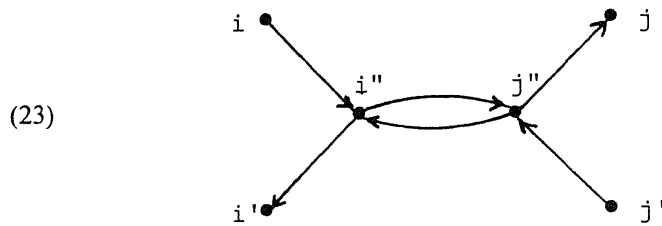
Moreover, (V, \bar{A}) is strongly connected. This follows directly from (20) if $(i,j) \notin A$. If $(i,j) \in A$, then $(i,i''), (i'',j''), (j'',j)$ form a path in \bar{A} from i to j . Hence also in this case, (V, \bar{A}) is strongly connected.

Conversely, let \bar{A} be a minimum length subset of $\bar{V} \times \bar{V}$ with (\bar{V}, \bar{A}) strongly connected. Without loss of generality, if $l(a,b) = 0$ then $(a,b) \in \bar{A}$. Let:

$$(22) \quad \begin{aligned} A &:= \bar{A} \cap (V \times V) && \text{if } (i'', j'') \notin \bar{A}, \\ A &:= (\bar{A} \cap (V \times V)) \cup \{(i,j)\} && \text{if } (i'', j'') \in \bar{A}. \end{aligned}$$

Again (21) holds. Moreover (V, A) is strongly connected. To see this, take

$a, b \in V$. We show that A contains a path from a to b . Since (\bar{V}, \bar{A}) is strongly connected, \bar{A} contains a path P from a to b . Assume that P passes i'' and j'' as few as possible. If P does not traverse i'' nor j'' , it is also a path in A . So suppose P traverses i'' or j'' . Consider all arcs incident to i'' or j'' with finite length:



Since $(i, i'), (j', i'), (j', j) \in A$, and since $(i, j) \in A$ if $(i'', j'') \in \bar{A}$, it follows that P does not intersect $\{i'', j''\}$.

So replacing V, l by \bar{V}, \bar{l} gives an equivalent problem, and decreases the number of pairs (i, j) with $0 < l(i, j) < \infty$ and $l(j, i) \neq 0$. Therefore, after at most $|V|^2$ such replacements, we attain an equivalent strong connectivity problem satisfying Lucchesi's condition. This is solvable in polynomial time by Lucchesi's algorithm. \square

This theorem suggests that in a branch and bound process, our branching strategy should strive for a situation where the following holds:

(24) for all $i, j \in V$: $(i, j) \in F$, or $(i, i'), (j', i'), (j', j) \in R$ for some $i', j' \in V$.

(The second alternative includes the case $(i, j) \in R$, by taking $i' = j, j' = i$.)

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